WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2019
TEST 4: Integration

Name: SOLUTIONS

Wednesday 3rd July 2019

Time: 50 minutes

Total marks: $\frac{}{25}+\frac{}{25}=\frac{}{50}$

Calculator free section – maximum 25 minutes

1. [6 marks – 4 and 2]

(a) The rational expression $\dfrac{3x-2}{x^2-3x+2}$ can be expressed in the form $\dfrac{A}{x+a}+\dfrac{B}{x+b}$.
Identify a suitable set of values for a, b, A and B.

$$= \frac{3x-2}{(x-2)(x-1)} \checkmark = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)} \checkmark$$

$$\Rightarrow \left.\begin{aligned} A+B &= 3 \\ -A-2B &= -2 \checkmark \end{aligned}\right\} \quad -B = 1 \quad , \quad B=-1, \; A=4 \checkmark$$

$\therefore$ $a=-2$ $\quad b=-1$

$A=4$ $\quad B=-1$ $\quad$ (or vice versa)

(b) Determine $\displaystyle\int \frac{3x-2}{x^2-3x+2}\,dx$

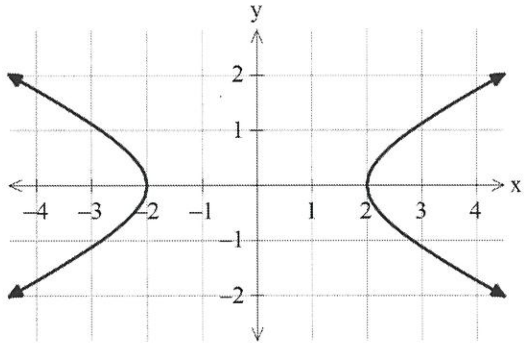
$$= \int \frac{4}{x-2} - \frac{1}{x-1}\,dx \checkmark$$

$$= 4\ln|x-2| - \ln|x-1| + c \qquad \text{or} \qquad \ln\left|\frac{(x-2)^4}{(x-1)}\right| + c$$

$\checkmark$

2. [5 marks]

The rectangular hyperbola $\frac{x^2}{4} - y^2 = 1$, as shown, is used as a model for the nose of a space craft.



Determine the exact volume generated when $\frac{x^2}{4} - y^2 = 1$ between $x = 2$ and $x = 4$ is revolved around the x axis.

$V = \pi \int_2^4 y^2 \, dx \quad = \pi \int_2^4 \frac{x^2}{4} - 1 \, dx$ ✓✓

$= \pi \left(\frac{x^3}{12} - x \right) \Big|_2^4$ ✓

$= \pi \left(\frac{64}{12} - 4 - \frac{8}{12} + 2 \right)$ ✓

$= \frac{8\pi}{3}$ units^3 ✓

$\frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

$- 2$

$= \frac{8}{3}$.

3. [4 marks]

Use the substitution $u = \ln x$ to evaluate $\int_1^e \frac{\ln x}{x} \, dx$

✓ limits

$u = \ln x$

$du = \frac{1}{x} dx$ ✓

$= \int_0^1 u \, du$ ✓

$= \frac{u^2}{2} \Big|_0^1$

$= \frac{1}{2}$ ✓

4. [10 marks – 3, 3, 2, and 2]

Calculate each of the following. The use of a substitution is optional.

(a) $\int \frac{x^2-3}{\sqrt{x^3-9x}}\,dx$

$= \frac{2}{3}\sqrt{x^3-9x} + C$ ✓ ✓ ✓

OR $u = x^3 - 9x$

$du = 3x^2 - 9\,dx = 3(x^2-3)\,dx$ ✓

So $\int = \int \frac{1}{3} u^{-\frac{1}{2}}\,du$ ✓

$= \frac{2}{3} u^{\frac{1}{2}} + C = \frac{2}{3}\sqrt{x^3-9x} + C$ ✓

(b) $\int 4\cos^3\theta\, d\theta$ (Put $u = \sin\theta$)

$= \int 4\cos\theta\,(1-\sin^2\theta)\,d\theta$ ✓

$= \int 4\cos\theta - 4\cos\theta.\sin^2\theta\,d\theta$ ✓

$= 4\sin\theta - \frac{4}{3}\sin^3\theta + C$ ✓

OR $du = \cos\theta\,d\theta$ ✓

$\int = \int 4\cos^2\theta\,du$

$= \int 4(1-u^2)\,du$ ✓

$= 4u - \frac{4u^3}{3} + C$

$= 4\sin\theta - \frac{4\sin^3\theta}{3} + C$ ✓

(c) $\int \sec^2 x \tan^2 x\, dx$ (Put $t = \tan x$)

$= \frac{\tan^3 x}{3} + C$ ✓✓ $\left(\text{since } \frac{d}{dx}\tan x = \sec^2 x\right)$

OR $dt = \sec^2 x\,dx$

$\int = \int t^2\,dt$ ✓

$= \frac{\tan^3 x}{3} + C$ ✓

(d) $\int \sec x \tan x\, dx$

$= \int \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\,d\theta$

$= \int \sin\theta.(\cos\theta)^{-2}\,d\theta$ ✓

$= (\cos\theta)^{-1} + C$

$= \frac{1}{\cos\theta} + C$ ✓ or $\sec\theta + C$

Working space:

Year 12 Specialist Test 4: Integrals

Name: ______________________

Time: 25 minutes

25 marks

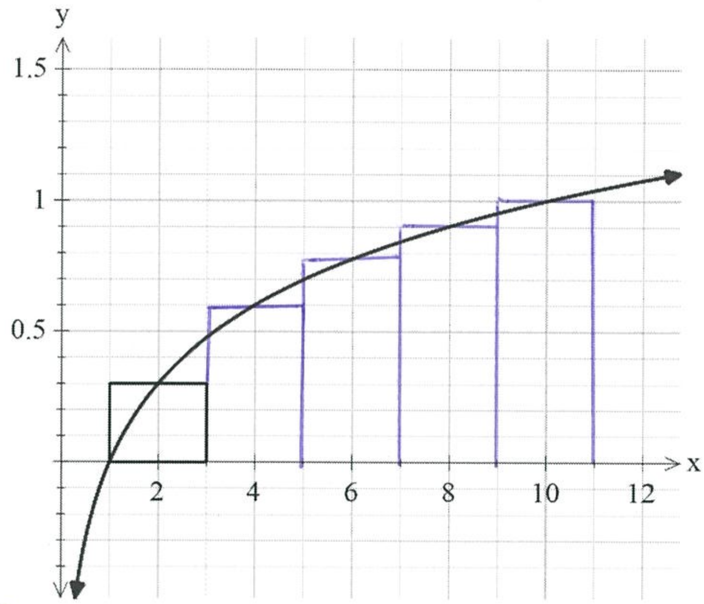
Calculator assumed section

5. [5 marks – 3 and 2]

The interval $1 \le x \le 11$ can be divided into 5 sub-intervals each of width 2.

(a) Use such a sub-division and the mid-point rectangle method to estimate $\int_1^{11} \log_{10} x \, dx$ to an accuracy of 3 decimal places.

The first rectangle is drawn.



Other rectangles drawn or implied ✓

$$\int_1^{11} \log_{10} x \, dx \simeq 2\log_{10} 2 + 2\log_{10} 4 + 2\log_{10} 6 + 2\log_{10} 8 + 2\log_{10} 10$$ ✓

$$= 2\log(2\times 4\times 6\times 8) + 2$$

$$= 7.169 \quad (3\text{ dp})$$ ✓

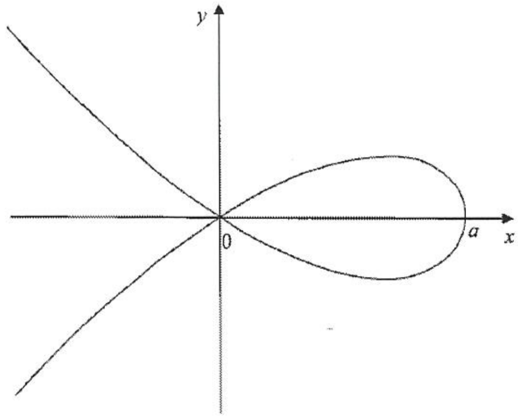
(b) What is the percentage error in this estimate?

$$\%\text{ error} = \frac{7.1687 - 7.1124}{7.1124} \doteq 0.79\%$$ ✓ ✓

ClassPad gives $\int_1^{11} \log_{10} x \, dx$ as 7.112374718.

6. [4 marks – 2 and 2]

This curve is an example of a right strophoid, with equation $y^2 = x^2(a-x)$, provided $a > 0$



(a) Express the area of the closed loop as an integral.

$$2\int_0^a x\sqrt{a-x}\ dx$$ ✓✓

(b) Simplify this integral (to an algebraic expression in terms of a)

$= \frac{8}{15} a^{\frac{5}{2}}$ units^2 ✓✓ or $\frac{8\sqrt{a^5}}{15}$ (ClassPad)

7. [3 marks]

The ante-derivative of $f(x) = \sin x \cos x$ can be found in three different ways:

(a) $\int \sin x \cos x\, dx = \int \frac{1}{2}\sin 2x\, dx$ since $\sin 2x = 2\sin x \cos x$

$$= -\frac{\cos 2x}{4} + C$$

(b) $\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + C$ since $\frac{d}{dx}(\sin x) = \cos x$

(c) $\int \sin x \cos x\, dx = -\frac{\cos^2 x}{2} + C$ since $\frac{d}{dx}(\cos x) = -\sin x$

Which of these three is correct? Justify your response.

All correct. ✓

$\cos 2x = 2\cos^2 x - 1$ ✓ $= 1 - 2\sin^2 x$ allows 3 different forms of same answer, with different C's ✓

8. [9 marks –2, 1, 2, 1, 2 and 1]

Use the behavior of the graphs $f(x)=x^2-1$, $g(x)=4^x$ and $h(x)=3x+1$ to:

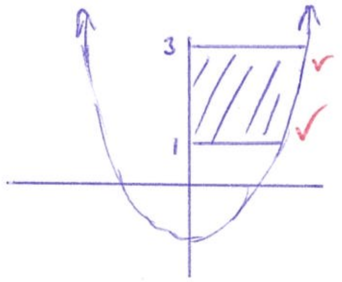
(a) describe the area represented by $\int_0^2 (x^2-1)\,dx$

Nett area, after $\int_0^1 < 0$ & $\int_1^2 > 0$

(b) write an integral to give the area enclosed between the graph of $y=f(x)$ and the x axis, from $x=0$ to $x=2$

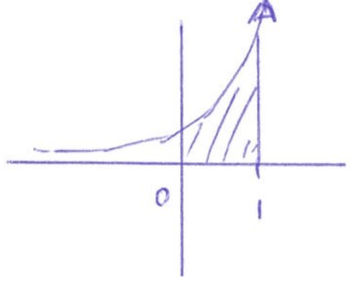
$$\int_0^2 |x^2-1|\,dx$$

(c) describe, or sketch, the area represented by $\int_1^3 \sqrt{y+1}\,dy$



area between $y=x^2-1$, y axis, $y=1$, $y=3$

(d) describe, or sketch, the shape represented by $2\pi\int_0^1 x\times 4^x\,dx$



Volume generated by rotation about y axis of region under the curve.

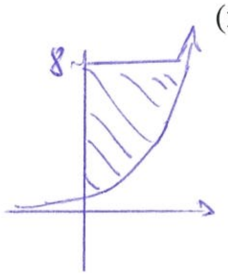
(e) write an integral to calculate the volume generated when the region enclosed by $g(x)=4^x$ and $h(x)=3x+1$ is revolved around the x axis

$$V = \pi\int_0^1 (3x+1)^2 - 4^{2x}\,dx$$

✓ $\int$

✓ squared first

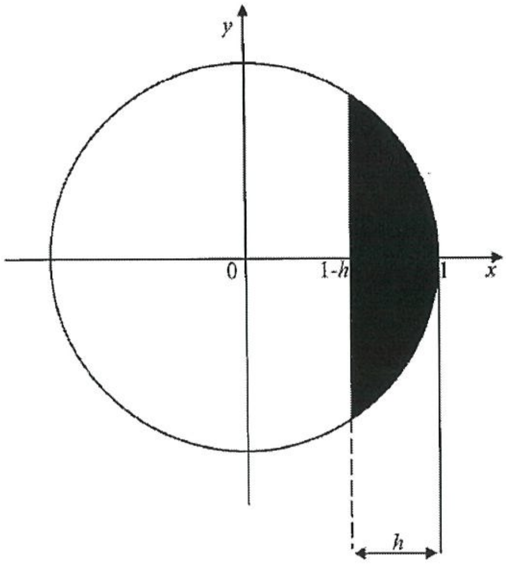
(f) describe, or sketch, the shape represented by $\pi\int_1^8 \log_4 y\,dy$



Volume of revolution about y axis between curve, y axis, $y=1$ and $y=8$

9. [4 marks –1 and 3]

This diagram shows a spherical cap of thickness h, generated by revolving part of the ~~sphere~~ $x^2 + y^2 = 1$ around the x axis.
circle



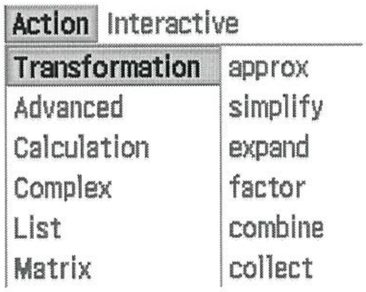
(a) Write down an integral to represent the volume of such a spherical cap.

$$\pi \int_{1-h}^{1} 1 - x^2 \, dx$$

✓ integral

(b) Show that this volume is $\frac{1}{3}\pi h^2(3-h)$.

(Some of the ClassPad operations illustrated may be helpful.)



$$= -\left(\frac{(h-1)^3}{3} - h + \frac{1}{3}\right)\pi$$ ✓

(ClassPad)

Simplify ans $$= \frac{-h^2(h-3)\pi}{3}$$ ✓

$$= \frac{1}{3}\pi h^2(3-h)$$